

# Observables and correlation functions in $OSp$ invariant string field theory

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**ABSTRACT:** We define BRST invariant observables in the  $OSp$  invariant closed string field theory for bosonic strings. We evaluate correlation functions of these observables and show that the S-matrix elements derived from them coincide with those of the light-cone gauge string field theory.

**KEYWORDS:** BRST Symmetry, String Field Theory, Bosonic Strings.

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## 1. Introduction

The  $OSp$  invariant string field theory [1–3] is a covariantized version of the light-cone gauge string field theory [4–6]. It is made to reproduce the results of light-cone gauge string field theory via Parisi-Sourlas mechanism [7]. Involving extra time and length variables, the formulation of the theory is not similar to the usual ones but rather like stochastic quantization. For noncritical strings, stochastic type formulation of string field theory was proposed to reproduce the results of matrix models [8]. Therefore it is likely that what can be done for noncritical strings can be done also for critical strings by using this  $OSp$  invariant string field theory formulation. Indeed, using the results of noncritical string theories [9, 10] and idempotency of boundary states [11, 12], we constructed solitonic operators which can be regarded as D-branes in the  $OSp$  invariant string field theory in [13].

Therefore the  $OSp$  invariant string field theory may be useful to study the nonperturbative effects involving D-branes. However, since the structure of the action is quite different from that of usual string field theories, it is not easy to see how the closed string particle modes are realized in this string field theory. What we would like to do in this paper is to clarify this point. We will consider the  $OSp$  invariant string field theory for closed bosonic strings and define BRST invariant observables corresponding to these particle modes and study their correlation functions. We will show that the S-matrix elements can be derived from the correlation functions and they coincide with those of the light-cone gauge string field theory.

Our treatment is different from the previous ones [2, 3] in which on-shell physical states are considered. Since the kinetic term of the action is not similar to that of the usual formulation, it is difficult to fix the normalization of these states.<sup>1</sup> By considering the observables instead, we can fix the normalization using the two-point correlation functions. Another advantage of our method is that we can show the Parisi-Sourlas reduction without Euclideanizing the  $\pm$  directions. The  $OSp$  invariant string field theory is inherently Lorentzian in these directions and the BRST cohomology is defined for such signature. Therefore it is important to show that the reduction occurs without changing the signature.

The organization of this paper is as follows. In section 2, we will review the  $OSp$  invariant string field theory. In section 3, we will define the observables of the  $OSp$  invariant string field theory. In section 4, we will study the correlation functions of the observables defined in section 3, and show that the S-matrix elements which can be derived from these correlation functions coincide with those in the light-cone gauge string field theory. Section 5 will be devoted to discussions.

In this paper, we set the string slope parameter  $\alpha'$  to be 2.

## 2. $OSp$ invariant string field theory

In order to fix the notations, we review the  $OSp$  invariant string field theory.<sup>2</sup>

The procedure of [1] for covariantizing the light-cone gauge string field theory [4, 5] is to replace the  $O(24)$  transverse vector  $X^i$  ( $i = 1, \dots, 24$ ) by the  $OSp(26|2)$  vector  $X^M = (X^\mu, C, \bar{C})$ , where  $X^\mu = (X^i, X^{25}, X^{26})$  are Grassmann even and the ghost fields  $C$  and  $\bar{C}$  are Grassmann odd. The metric of the  $OSp(26|2)$  vector space is<sup>3</sup>

$$\eta_{MN} = \begin{array}{c} \begin{array}{c} C \quad \bar{C} \\ \left( \begin{array}{c|c} \delta_{\mu\nu} & \\ \hline 0 & -i \\ i & 0 \end{array} \right) \\ \begin{array}{c} C \\ \bar{C} \end{array} \end{array} \end{array} = \eta^{MN} . \quad (2.1)$$

In accordance with the above  $OSp$  extension, we extend the oscillation modes  $\alpha_n^i$  and  $\tilde{\alpha}_n^i$  ( $i = 1, \dots, 24; n \in \mathbb{Z}$ ) as well in the following way,

$$\begin{aligned} x^i &\longrightarrow x^M = (x^\mu, C_0, \bar{C}_0) , \\ \alpha_0^i = \tilde{\alpha}_0^i = p^i &\longrightarrow \alpha_0^M = \tilde{\alpha}_0^M = p^M = (p^\mu, -\pi_0, \bar{\pi}_0) , \\ \alpha_n^i &\longrightarrow \alpha_n^M = (\alpha_n^\mu, -\gamma_n, \bar{\gamma}_n) , \quad \tilde{\alpha}_n^i \longrightarrow \tilde{\alpha}_n^M = (\tilde{\alpha}_n^\mu, -\tilde{\gamma}_n, \tilde{\bar{\gamma}}_n) \quad \text{for } n \neq 0 . \end{aligned} \quad (2.2)$$

<sup>1</sup>In [3], this problem was addressed, and solved for covariantized light-cone string field theory [14, 15].

<sup>2</sup>In this paper, we use conventions slightly different from those of our previous paper [13], in particular that for the integration measure of the momentum zero-modes.

<sup>3</sup>In this paper, we begin by the Euclidean signature for the metric in the linearly realized  $O(26)$  directions. This is different from the original formulation in [1] where the signature of the metric in these directions is Lorentzian.

These oscillators satisfy the canonical commutation relations

$$[x^N, p^M] = i\eta^{NM}, \quad [\alpha_n^N, \alpha_m^M] = n\eta^{NM}\delta_{n+m,0}, \quad [\tilde{\alpha}_n^N, \tilde{\alpha}_m^M] = n\eta^{NM}\delta_{n+m,0} \quad (2.3)$$

for  $n, m \neq 0$ , where the graded commutator  $[A, B]$  denotes the anti-commutator when  $A$  and  $B$  are both fermionic operators and the commutator otherwise.

We describe the Hilbert space for the string by the Fock space of the oscillators for the non-zero modes and the wave functions for the zero-modes. We take the momentum representation of the wave functions for the zero-modes  $p^\mu, \alpha, \pi_0, \bar{\pi}_0$ , where  $\alpha$  is identified with the string length. In this description, the vacuum state  $|0\rangle$  in the first quantization is defined by

$$x^M|0\rangle = i\eta^{MN}\frac{\partial}{\partial p^N}|0\rangle = 0, \quad \alpha_n^M|0\rangle = \tilde{\alpha}_n^M|0\rangle = 0 \quad \text{for } n > 0. \quad (2.4)$$

The integration measure for the zero-modes of the  $r$ -th string is defined as

$$dr \equiv \frac{\alpha_r d\alpha_r}{2} \frac{d^{26}p_r}{(2\pi)^{26}} i d\bar{\pi}_0^{(r)} d\pi_0^{(r)}. \quad (2.5)$$

The action of the  $OSp$  invariant string field theory is obtained by the  $OSp$  extension explained above from that of the light-cone gauge string field theory given in [4]. This takes the form

$$S = \int dt \left[ \frac{1}{2} \int d1d2 \langle R(1,2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - \frac{L_0^{(2)} + \tilde{L}_0^{(2)} - 2}{\alpha_2} \right) | \Phi \rangle_2 \right. \\ \left. + \frac{2g}{3} \int d1d2d3 \langle V_3^0(1,2,3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right], \quad (2.6)$$

where  $\langle R(1,2) |$  is the reflector defined in eq. (A.1) and  $\langle V_3^0(1,2,3) |$  is the three-string vertex given as

$$\langle V_3^0(1,2,3) | \equiv \delta(1,2,3) {}_{123}\langle 0 | e^{E(1,2,3)} \mathcal{P}_{123} \frac{|\mu(1,2,3)|^2}{\alpha_1 \alpha_2 \alpha_3}. \quad (2.7)$$

$E(1,2,3)$ ,  $\mathcal{P}_{123}$ ,  $\delta(1,2,3)$  and  $\mu(1,2,3)$  in this equation are defined in eq. (A.5). The string field  $\Phi$  is taken to be Grassmann even and subject to the level matching condition  $\mathcal{P}\Phi = \Phi$  and the reality condition

$$\langle \Phi_{\text{hc}} | = \langle \Phi |, \quad (2.8)$$

where  $\langle \Phi_{\text{hc}} | \equiv (|\Phi\rangle)^\dagger$  denotes the hermitian conjugate of  $|\Phi\rangle$ , and  $\langle \Phi |$  denotes the BPZ conjugate of  $|\Phi\rangle$  defined in eq. (A.3).

The action (2.6) is invariant under the BRST transformation

$$\delta_B \Phi = Q_B \Phi + g \Phi * \Phi, \quad (2.9)$$

where the  $*$ -product is given in eq. (A.6) and the BRST operator  $Q_B$  is defined [16, 17] as

$$Q_B = \frac{C_0}{2\alpha} (L_0 + \tilde{L}_0 - 2) - i\pi_0 \frac{\partial}{\partial \alpha} \\ + \frac{i}{\alpha} \sum_{n=1}^{\infty} \left( \frac{\gamma_{-n} L_n - L_{-n} \gamma_n}{n} + \frac{\tilde{\gamma}_{-n} \tilde{L}_n - \tilde{L}_{-n} \tilde{\gamma}_n}{n} \right). \quad (2.10)$$

Here  $L_n$  and  $\tilde{L}_n$  ( $n \in \mathbb{Z}$ ) are the Virasoro generators given by

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \circ \alpha_{n+m}^N \alpha_{-m}^M \eta_{NM} \circ, \quad \tilde{L}_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \circ \tilde{\alpha}_{n+m}^N \tilde{\alpha}_{-m}^M \eta_{NM} \circ, \quad (2.11)$$

where the symbol  $\circ \circ$  denotes the normal ordering of the oscillators in which the non-negative modes should be placed to the right of the negative modes. The BRST operator (2.10) can be identified with the  $M^{C-}$  element of the  $OSp(27, 1|2)$  Lorentz generators [16, 17].

### 3. BRST cohomology and observables

In the  $OSp$  invariant string field theory, we consider BRST invariant objects as physical quantities. The S-matrix elements are defined for on-shell BRST invariant states. In this section, we will first show that these states correspond to the on-shell physical states of string theory. Then we define observables from whose correlation functions we can deduce the S-matrix elements of the  $OSp$  invariant theory.

#### 3.1 BRST cohomology of $Q_B$

In order to obtain the BRST cohomology of the asymptotic states in the  $OSp$  invariant string field theory, we need the BRST cohomology of the operator  $Q_B$ . For studying the BRST cohomology of  $Q_B$ , it is convenient to relate  $Q_B$  to Kato-Ogawa's BRST operator [18]. The worldsheet variables in the ghost sector of the  $OSp$  invariant string field theory can be identified with the  $(b, c)$  ghost variables as

$$\begin{aligned} C_0 &= 2\alpha c_0^+, & \bar{\pi}_0 &= \frac{1}{2\alpha} b_0^+, \\ \gamma_n &= i n \alpha c_n, & \tilde{\gamma}_n &= i n \alpha \tilde{c}_n, & \bar{\gamma}_n &= \frac{1}{\alpha} b_n, & \tilde{\bar{\gamma}}_n &= \frac{1}{\alpha} \tilde{b}_n \end{aligned} \quad (3.1)$$

for  $n \neq 0$ . From this identification, one can find that the  $OSp$  invariant string field theory includes extra variables  $\pi_0, \alpha$  besides those in the usual covariantly quantized theory. Here let us introduce  $c \equiv \frac{\pi_0}{\alpha}$  for later convenience. Then the first-quantized Hilbert space of the  $OSp$  invariant theory is the tensor product of that of the usual covariant string theory and that of  $c, \alpha$ . The BRST operator can be written as

$$Q_B = Q_B^{KO} - ic \left( \alpha \frac{\partial}{\partial \alpha} \Big|_{b,c} + 1 \right), \quad (3.2)$$

where  $Q_B^{KO}$  is the usual Kato-Ogawa's BRST operator with  $b_0^-$  omitted and  $\frac{\partial}{\partial \alpha} \Big|_{b,c}$  denotes the derivative with  $c, c_0^+, b_0^+, c_n, b_n, \tilde{c}_n, \tilde{b}_n (n \neq 0)$  kept fixed.

Now let us see what a BRST closed state  $|\rangle$  looks like. It is convenient to expand  $|\rangle$  in  $c$ :

$$|\rangle = |1\rangle + c|2\rangle, \quad (3.3)$$

where the states  $|1\rangle$  and  $|2\rangle$  are independent of  $c$ . In this notation, the condition that  $|\rangle$  is  $Q_B$ -closed becomes

$$Q_B^{KO}|1\rangle = 0, \quad (3.4)$$

$$Q_B^{KO}|2\rangle = \mathcal{D}_\alpha|1\rangle, \quad (3.5)$$

where

$$\mathcal{D}_\alpha \equiv -i \left( \alpha \frac{\partial}{\partial \alpha} \Big|_{b,c} + 1 \right). \quad (3.6)$$

Since we know the BRST cohomology of  $Q_B^{\text{KO}}$ , solutions to eq. (3.4) can be easily found to be a linear combination of states of the form  $f(\alpha)|\text{phys}\rangle$  and  $g(\alpha)Q_B^{\text{KO}}|\gamma'\rangle$ , where  $|\text{phys}\rangle$  denotes a state in a nontrivial cohomology class of  $Q_B^{\text{KO}}$  and  $f(\alpha), g(\alpha)$  are arbitrary functions of  $\alpha$ . Substituting these into eq. (3.5), one can see  $f(\alpha) = \frac{1}{\alpha}$  and  $|2\rangle$  should be a linear combination of the solutions to

$$Q_B^{\text{KO}}|2\rangle = (\mathcal{D}_\alpha g(\alpha))Q_B^{\text{KO}}|\gamma'\rangle. \quad (3.7)$$

Solutions to this equation can also be easily found and eventually we see that the BRST closed state  $|\gamma\rangle$  should be a linear combination of the states of the form

$$\frac{1}{\alpha}|\text{phys}\rangle, \quad (3.8)$$

and

$$ch(\alpha)|\text{phys}\rangle, \quad (3.9)$$

up to  $Q_B$  exact states. Here  $h(\alpha)$  is an arbitrary function of  $\alpha$ .

For  $|\text{phys}\rangle$ , one can choose the states of the form

$$|0\rangle_{b,c} \otimes \overline{|\text{primary}\rangle}_X (2\pi)^{26} \delta(p-k), \quad (3.10)$$

or

$$b_0^+ |0\rangle_{b,c} \otimes \overline{|\text{primary}\rangle}_X (2\pi)^{26} \delta(p-k), \quad (3.11)$$

where  $|0\rangle_{b,c}$  is a vacuum for the  $(b,c)$  ghosts satisfying  $c_0^+ |0\rangle_{b,c} = 0$ .<sup>4</sup>  $\overline{|\text{primary}\rangle}_X$  is the oscillator part of a Virasoro primary state in the Hilbert space of  $X^\mu$  variables and  $k^\mu$  is the momentum eigenvalue. In order for these states to be BRST closed and nontrivial, the conformal weight of the the primary state  $\overline{|\text{primary}\rangle}_X (2\pi)^{26} \delta(p-k)$  should be  $(1,1)$ . This condition can be regarded as the on-shell condition for the particle corresponding to this string state.

**$\alpha$ -dependence.** The wave functions for the  $OSp$  invariant string field theory should satisfy appropriate boundary conditions. Especially one should be careful about the dependence on the zero-mode  $\alpha$ . Treating the regions  $\alpha > 0$  and  $\alpha < 0$  separately, let us introduce a real variable  $\omega$  as

$$\alpha = \pm e^\omega. \quad (3.12)$$

If we express the wave functions using the original variables in the  $OSp$  invariant string field theory, the  $\alpha$  dependent part of the wave functions should be of the form

$$e^{-\omega} f(\omega), \quad (3.13)$$

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<sup>4</sup>Notice that  $b_0^-, c_0^-$  are omitted.

where  $f(\omega)$  is a delta function normalizable function with respect to the norm

$$\|f\|^2 \equiv \int_{-\infty}^{\infty} d\omega |f(\omega)|^2 . \quad (3.14)$$

We can take  $e^{i\omega x}$  ( $x \in \mathbb{R}$ ) as a basis for such wave functions. It is straightforward to show that under such conditions  $Q_B$  and  $M^{+-}$  are hermitian. If we express the wave functions using the  $(b, c)$  ghosts,  $\alpha$  and  $c$ , eq. (3.13) should be replaced by

$$e^{(n-1)\omega} f(\omega) , \quad (3.15)$$

where  $n$  is the ghost number of the state. The ghost number is defined so that the variable  $c$  has ghost number 1 and the state  $|0\rangle_{b,c}$  has ghost number 0.

Now let us take this condition into account and further restrict the form of the BRST closed states. For the states of the form in eq. (3.8),  $|\text{phys}\rangle$  should have ghost number 0 and therefore it should be of the form

$$\frac{1}{\alpha} |0\rangle_{b,c} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) . \quad (3.16)$$

The states of the form in eq. (3.9) should be either

$$c e^{i\omega x} |0\rangle_{b,c} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.17)$$

or

$$\frac{b_0^+ c}{\alpha} e^{i\omega x} |0\rangle_{b,c} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.18)$$

but the former one is BRST exact and the latter is BRST exact if  $x \neq 0$ . Therefore we have shown that the BRST closed states can be written as a linear combination of the states of the form

$$\frac{1}{\alpha} |0\rangle_{b,c} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.19)$$

and

$$\frac{b_0^+ c}{\alpha} |0\rangle_{b,c} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.20)$$

up to BRST exact states. Written in terms of the original variables of the  $OSp$  theory, these are

$$\frac{1}{\alpha} |0\rangle_{C, \bar{C}} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.21)$$

and

$$\frac{1}{\alpha} \bar{\pi}_0 \pi_0 |0\rangle_{C, \bar{C}} \otimes |\overline{\text{primary}}\rangle_X (2\pi)^{26} \delta(p - k) , \quad (3.22)$$

where  $|0\rangle_{C, \bar{C}}$  is the oscillator vacuum (2.4) for the  $C, \bar{C}$  sector.

### 3.2 Observables

In order to deal with the BRST invariant asymptotic states of the  $OSp$  invariant string field theory, we define the BRST invariant observables corresponding to them. They are of the form

$$\mathcal{O} = \langle |\Phi\rangle . \quad (3.23)$$

Here  $|\rangle$  is a first quantized string state and the inner product should be considered as including the integrations in the zero-mode part. The BRST transformation of this quantity is given as

$$\delta_B \mathcal{O} = \langle | \left( Q_B |\Phi\rangle + g |\Phi * \Phi\rangle \right) . \quad (3.24)$$

In the discussion of the asymptotic states, the second term in the transformation can be ignored. Therefore for BRST invariant states, we should impose the condition

$$\langle | Q_B |\Phi\rangle = 0 , \quad (3.25)$$

which implies

$$Q_B |\rangle = 0 . \quad (3.26)$$

For a BRST exact state  $|\rangle = Q_B |\rangle'$ ,

$$\mathcal{O} \simeq \delta_B \langle |\Phi\rangle , \quad (3.27)$$

up to multi-string contribution. Therefore  $|\rangle$  should be chosen from a nontrivial cohomology class of  $Q_B$ , which was given in the previous subsection.

The string field  $|\Phi\rangle$  can be expanded in terms of  $\pi_0$  and  $\bar{\pi}_0$  as

$$|\Phi\rangle = |\bar{\phi}\rangle + i\pi_0 |\bar{\chi}\rangle + i\bar{\pi}_0 |\chi\rangle + i\pi_0 \bar{\pi}_0 |\phi\rangle . \quad (3.28)$$

Substituting this into the kinetic term of the action eq. (2.6), the only term which is quadratic in  $\bar{\phi}$  is

$$\int dt \int d\alpha \frac{d^{26}p}{(2\pi)^{26}} \frac{1}{2} \langle \bar{\phi} | \bar{\phi} \rangle . \quad (3.29)$$

The interaction terms are at most quadratic in  $\bar{\phi}$ . Therefore  $\bar{\phi}$  can be regarded as an auxiliary field and integrated out. We identify  $\phi$  with the usual physical closed string modes. If one integrates  $\bar{\phi}$  out, the kinetic term for  $\phi$  looks quite different from that of the usual field theory. It rather looks similar to that of the stochastic quantization.

Thus for constructing the observables, we discard the case when  $\langle |\Phi\rangle$  is an auxiliary mode. Then  $|\rangle$  should be of the form eq. (3.21). Making the zero-mode integral explicit, one can describe the observables constructed above as

$$\mathcal{O}(t, k) = \frac{i}{2} \int_{-\infty}^{\infty} d\alpha \int d\bar{\pi}_0 d\pi_{0, \bar{C}} \langle 0 | \otimes_X \overline{\text{primary}} | \Phi(t, \alpha, \pi_0, \bar{\pi}_0, k) \rangle . \quad (3.30)$$

Here the integration measure in eq. (2.5) is used for the zero-mode integration. We normalize the state  $\overline{\text{primary}}|_X$  so that

$${}_X \overline{\text{primary}} \overline{\text{primary}}|_X = 1 . \quad (3.31)$$



If the state  $|\overline{\text{primary}}\rangle_X \otimes |0\rangle_{C,\bar{C}} (2\pi)^{26} \delta^{26}(p-k)$  satisfies the relation

$$\begin{aligned} \left( L_0 + \tilde{L}_0 - 2 \right) |\overline{\text{primary}}\rangle_X \otimes |0\rangle_{C,\bar{C}} (2\pi)^{26} \delta^{26}(p-k) = \\ (k^2 + 2i\pi_0 \bar{\pi}_0 + M^2) |\overline{\text{primary}}\rangle_X \otimes |0\rangle_{C,\bar{C}} (2\pi)^{26} \delta^{26}(p-k), \end{aligned} \quad (3.32)$$

this state is to be considered as a particle state with mass  $M$ .  $\mathcal{O}(t, k)$  is BRST exact unless  $k^2 + M^2 = 0$ .

### 3.3 Free propagator

We would like to study BRST invariant asymptotic states using the observables constructed above. Once the auxiliary field  $\bar{\phi}$  is integrated out, the action no longer possesses the kinetic term similar to that of the usual field theory action. Therefore it may seem unlikely that these observables correspond to usual particle states. However, as we will show, the free propagators corresponding to these operators yield propagators for particles propagating in 26 dimensions.

Let us consider the observables  $\mathcal{O}_r(t_r, p_r)$  ( $r = 1, 2$ ) which are of the form eq. (3.30) corresponding to a common primary state, i.e.  $|\overline{\text{primary}}\rangle_X \equiv |\overline{\text{primary}}_1\rangle_X = |\overline{\text{primary}}_2\rangle_X$  and  $M \equiv M_1 = M_2$ . We define the two-point function

$$\left\langle\left\langle \tilde{\mathcal{O}}_1(E_1, p_1) \tilde{\mathcal{O}}_2(E_2, p_2) \right\rangle\right\rangle \equiv \int dt_1 dt_2 e^{iE_1 t_1 + iE_2 t_2} \langle\langle 0 | T \mathcal{O}_1(t_1, p_1) \mathcal{O}_2(t_2, p_2) | 0 \rangle\rangle, \quad (3.33)$$

where  $|0\rangle\rangle$  denotes the vacuum in the second quantization. The lowest order contribution can be written by using the Feynman propagator as

$$\prod_{r=1}^2 \left( \frac{i}{2} \int d\alpha_r d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \right) \frac{i\delta(1,2) 2\pi \delta(E_1 + E_2)}{\alpha_1 E_1 - p_1^2 - M^2 - 2i\pi_0^{(1)} \bar{\pi}_0^{(1)} + i\epsilon}, \quad (3.34)$$

where  $\delta(1,2)$  is given in eq. (A.2). In the string perturbation theory, the propagator corresponds to a cylindrical worldsheet and it is calculated as

$$\begin{aligned} \frac{i\delta(1,2) 2\pi \delta(E_1 + E_2)}{\alpha_1 E_1 - p_1^2 - M^2 - 2i\pi_0^{(1)} \bar{\pi}_0^{(1)}} = \int dt_1 dt_2 e^{iE_1 t_1 + iE_2 t_2} \frac{1}{|\alpha_1|} \delta(1,2) \times \\ \times \left[ \theta(\alpha_1) \theta(t_1 - t_2) e^{-i\frac{t_1 - t_2}{\alpha_1} (p_1^2 + 2i\pi_0^{(1)} \bar{\pi}_0^{(1)} + M^2)} \right. \\ \left. + \theta(\alpha_2) \theta(t_2 - t_1) e^{-i\frac{t_2 - t_1}{\alpha_2} (p_2^2 + 2i\pi_0^{(2)} \bar{\pi}_0^{(2)} + M^2)} \right] \end{aligned} \quad (3.35)$$

Let us use this expression to evaluate eq. (3.34). Substituting eq. (3.35) into eq. (3.34), we obtain

$$\begin{aligned} \left\langle\left\langle \tilde{\mathcal{O}}_1(E_1, p_1) \tilde{\mathcal{O}}_2(E_2, p_2) \right\rangle\right\rangle_{\text{free}} = \int dt_1 dt_2 e^{iE_1 t_1 + iE_2 t_2} (2\pi)^{26} \delta^{26}(p_1 + p_2) \\ \times \left[ \theta(t_1 - t_2) \frac{i}{2} \int_0^\infty \frac{d\alpha_1}{\alpha_1} \int d\bar{\pi}_0^{(1)} d\pi_0^{(1)} e^{-i\frac{t_1 - t_2}{\alpha_1} (p_1^2 + M^2 + 2i\pi_0^{(1)} \bar{\pi}_0^{(1)})} \right. \\ \left. + \theta(t_2 - t_1) \frac{i}{2} \int_0^\infty \frac{d\alpha_2}{\alpha_2} \int d\bar{\pi}_0^{(2)} d\pi_0^{(2)} e^{-i\frac{t_2 - t_1}{\alpha_2} (p_2^2 + M^2 + 2i\pi_0^{(2)} \bar{\pi}_0^{(2)})} \right]. \end{aligned} \quad (3.36)$$

The integrations over  $\alpha$ ,  $\pi_0$  and  $\bar{\pi}_0$  can be done as

$$\begin{aligned} \frac{i}{2} \int_0^\infty \frac{d\alpha_1}{\alpha_1} \int d\bar{\pi}_0^{(1)} d\pi_0^{(1)} e^{-i\frac{t_1-t_2}{\alpha_1}(p_1^2+M^2+2i\pi_0^{(1)}\bar{\pi}_0^{(1)}-i\epsilon)} &= \int_0^\infty \frac{d\alpha_1}{\alpha_1} i \frac{t_1-t_2}{\alpha_1} e^{-i\frac{t_1-t_2}{\alpha_1}(p_1^2+M^2-i\epsilon)} \\ &= i \int_0^\infty dt e^{-it(p_1^2+M^2-i\epsilon)} = \frac{1}{p_1^2+M^2}, \end{aligned} \quad (3.37)$$

where we introduced  $t \equiv \frac{t_1-t_2}{\alpha_1}$ . Therefore we eventually get

$$\begin{aligned} \left\langle\left\langle \tilde{\mathcal{O}}_1(E_1, p_1) \tilde{\mathcal{O}}_2(E_2, p_2) \right\rangle\right\rangle_{\text{free}} &= \int dt_1 dt_2 e^{iE_1 t_1 + iE_2 t_2} (2\pi)^{26} \delta^{26}(p_1 + p_2) \left[ \frac{\theta(t_1 - t_2)}{p_1^2 + M^2} + \frac{\theta(t_2 - t_1)}{p_2^2 + M^2} \right] \\ &= \frac{(2\pi)^{26} \delta^{26}(p_1 + p_2)}{p_1^2 + M^2} 2\pi \delta(E_1) 2\pi \delta(E_2). \end{aligned} \quad (3.38)$$

The reason why we have factors  $2\pi\delta(E_r)$  can be understood as follows. As we can see from the expression eq. (3.35),  $\frac{L_0 + \tilde{L}_0 - 2}{\alpha}$  is the Hamiltonian on the worldsheet. Since this is a BRST exact operator,  $\mathcal{O}(t + \delta t, p)$  and  $\mathcal{O}(t, p)$  are BRST equivalent and only the constant mode with  $E = 0$  survives. Thus we here choose

$$\begin{aligned} \varphi(p) &\equiv \int \frac{dE}{2\pi} \tilde{\mathcal{O}}(E, p) \\ &= \mathcal{O}(t = 0, p), \end{aligned} \quad (3.39)$$

as a representative of these equivalent operators.

Hence we have

$$\left\langle\left\langle \varphi_1(p_1) \varphi_2(p_2) \right\rangle\right\rangle_{\text{free}} = \frac{(2\pi)^{26} \delta^{26}(p_1 + p_2)}{p_1^2 + M^2}. \quad (3.40)$$

This coincides with the Euclidean propagator for a particle with mass  $M$ . Thus we have shown that although the string field action possesses an unusual form, modes corresponding to the operators

$$\begin{aligned} \varphi(p) &= \frac{i}{2} \int_{-\infty}^\infty d\alpha \int d\bar{\pi}_0 d\pi_0 {}_{C, \bar{C}} \langle 0 | \otimes_X \langle \overline{\text{primary}} | \Phi(t = 0, \alpha, \pi_0, \bar{\pi}_0, p) \rangle \\ &= \frac{i}{2} \int \frac{dE}{2\pi} \int_{-\infty}^\infty d\alpha \int d\bar{\pi}_0 d\pi_0 {}_{C, \bar{C}} \langle 0 | \otimes_X \langle \overline{\text{primary}} | \tilde{\Phi}(E, \alpha, \pi_0, \bar{\pi}_0, p) \rangle, \end{aligned} \quad (3.41)$$

yield usual propagators. Here  $\tilde{\Phi}$  is the Fourier transform of  $\Phi$  with respect to  $t$ .  $\varphi(p)$  corresponds to a particle included in string theory.

For other modes, things are not so simple in general. For our purpose, it is necessary to check the two-point functions involving BRST exact observables  $\mathcal{O} = \langle |Q_B| \Phi \rangle$ . Notice that the free propagator

$$\left\langle\left\langle \langle |Q_B| \Phi \rangle \langle | \Phi \rangle \right\rangle\right\rangle_{\text{free}}, \quad (3.42)$$

is not necessarily 0, even if  $Q_B| \rangle' = 0$ . Indeed, calculating this quantity boils down to evaluating

$$\left( \langle |Q_B| \rangle \exp \left( -it \frac{L_0 + \tilde{L}_0 - 2 - i\epsilon}{\alpha} \right) | \right)' \quad (3.43)$$

which is 0 if we can make  $Q_B$  act on the state on the right. In order to do so, we should perform a partial integration over  $\alpha$ . If the integrand does not vanish for  $\alpha \rightarrow \infty$ , we have a nonvanishing surface term and obtain a nonvanishing result.<sup>5</sup> For example,  $\varphi(p)$  itself is actually BRST exact for  $p^2 + M^2 \neq 0$ , but we have eq. (3.40). Anyway, contributions for the correlation functions involving such BRST exact operators come from the boundary of the moduli space of the worldsheet, which is usual in string perturbation theory. Therefore, we do not expect to find particle poles such as  $\frac{1}{p^2 + M^2}$  in such correlators. Indeed it is straightforward to check that the correlation function eq. (3.42) with  $Q_B | \rangle' = 0$  does not yield such poles, provided  $| \rangle$  exists for  $p^2 + M^2 = 0$ .

Now that we identify the modes of  $\Phi$  corresponding to the particle states in string theory, we can construct the asymptotic states using them. Wick rotating as  $x^{26} \rightarrow x^0 = -ix^{26}$ , we can canonically quantize the theory considering  $x^0$  as time. Since the free propagator corresponding to  $\varphi(p)$  coincides with that for a particle with mass  $M$ , it is straightforward to define properly normalized asymptotic states using these operators.

We may be able to proceed and calculate the S-matrix elements for these asymptotic states. The calculations will be essentially the same as those in [3]. Using a generalization of the Parisi-Sourlas formula, we may be able to show that the S-matrix elements coincide with those of the light-cone gauge string field theory. However, in this paper we will rather calculate the correlation functions of the observables and define the S-matrix elements using them. By doing so, we can proceed without Wick rotating the  $\pm$  directions. Such a Wick rotation is necessary for deriving the Parisi-Sourlas type formula in this context [2, 3].

#### 4. Correlation functions and S-matrix elements

We have one observable  $\varphi(p)$  for one primary state in the Hilbert space of  $X^\mu$ . These primary states are in one-to-one correspondence to the states with physical polarizations in string theory. The correlation functions of the operators  $\varphi(p)$  can be considered as those in the 26 dimensional Euclidean space. Essentially, what we would like to show in this section is that these correlation functions can be considered as the correlation functions for the bosonic string field theory. We will prove that the S-matrix elements derived from these correlation functions coincide with those of the light-cone gauge string field theory.

##### 4.1 Correlation functions

Let us consider  $N$ -point correlation functions ( $N \geq 3$ )

$$\left\langle\left\langle \prod_{r=1}^N \varphi_r(p_r) \right\rangle\right\rangle, \tag{4.1}$$

of the observables

$$\varphi_r(p_r) = \int \frac{dE_r}{2\pi} \tilde{\mathcal{O}}_r(E_r, p_r) \quad (r = 1, \dots, N), \tag{4.2}$$

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<sup>5</sup>Notice that for the BRST exact operator  $\mathcal{O}(t + \delta t, p) - \mathcal{O}(t, p)$  we discussed above, this does not happen.

which are made from Virasoro primary states  $|\overline{\text{primary}}_r\rangle_X$  corresponding to particles with mass  $M_r$ . We will show that these correlation functions yield S-matrix elements for string theory. In order to evaluate this, we start from the correlation function

$$\left\langle\left\langle \prod_{r=1}^N \tilde{\mathcal{O}}_r(E_r, p_r) \right\rangle\right\rangle \equiv \prod_{r=1}^N \left( \int dt_r e^{iE_r t_r} \right) \langle\langle 0 | T \prod_{r=1}^N \mathcal{O}_r(t_r, p_r) | 0 \rangle\rangle. \quad (4.3)$$

The  $OSp$  invariant string field theory can be regarded as a light-cone gauge string field theory with “transverse” space-time coordinates  $X^M$  and the perturbative expansion can be obtained as in the usual light-cone gauge string field theory. The correlation function (4.3) can be calculated perturbatively and takes the form

$$\begin{aligned} \left\langle\left\langle \prod_{r=1}^N \tilde{\mathcal{O}}_r(E_r, p_r) \right\rangle\right\rangle &= \prod_{r=1}^N \left( \frac{i}{2} \int d\alpha_r d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \frac{i}{\alpha_r E_r - p_r^2 - M_r^2 - 2i\pi_0^{(r)} \bar{\pi}_0^{(r)}} \right) \\ &\times \delta^{OSp} \left( \sum_{s=1}^N p_s^{OSp} \right) G_{\text{amputated}}^{OSp}(p_1^{OSp}, \dots, p_N^{OSp}). \end{aligned} \quad (4.4)$$

Here  $p_r^{OSp}$  denotes the zero-modes  $p_r^{OSp} = (E_r, \alpha_r, p_r^\mu, \bar{\pi}_0^{(r)}, \pi_0^{(r)})$  of the  $r$ -th string and

$$\delta^{OSp} \left( \sum_{r=1}^N p_r^{OSp} \right) = 2\pi \delta \left( \sum_{r=1}^N E_r \right) \delta \left( \sum_{r=1}^N \alpha_r \right) (2\pi)^{26} \delta^{26} \left( \sum_{r=1}^N p_r \right) 2i \left( \sum_{r=1}^N \bar{\pi}_0^{(r)} \right) \left( \sum_{r=1}^N \pi_0^{(r)} \right). \quad (4.5)$$

$G_{\text{amputated}}^{OSp}(p_1^{OSp}, \dots, p_N^{OSp})$  is the amputated Green’s function, which is expressed as

$$G_{\text{amputated}}^{OSp}(p_1^{OSp}, \dots, p_N^{OSp}) = \sum_G \int \prod_I d\alpha_I \prod_n d(t_n - t_{n-1}) F_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}), \quad (4.6)$$

where  $G$  denotes a light-cone string diagram,  $\alpha_I$  denotes the string length for an internal line of the diagram and  $t_n$  denotes the proper time for a three-string vertex. Because of the conservation of the string length, some of  $\alpha_I$  can be expressed by other  $\alpha$ ’s through delta functions involved in  $F_G$ . Independent  $\alpha_I$ ’s and  $t_n - t_{n-1}$  can be regarded as the moduli of the Riemann surface corresponding to the light-cone string diagram. If none of the ratios  $\frac{\alpha_I}{\alpha_{I'}}$ ,  $\frac{t_n - t_{n-1}}{\alpha_I}$  are 0 or infinity, the diagram corresponds to a nondegenerate Riemann surface. Using the expression eq. (4.6), one can rewrite eq. (4.4) as

$$\left\langle\left\langle \prod_{r=1}^N \tilde{\mathcal{O}}_r(E_r, p_r) \right\rangle\right\rangle = \sum_G \int \prod_I d\alpha_I \prod_n d(t_n - t_{n-1}) I_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}), \quad (4.7)$$

where

$$\begin{aligned} I_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}) &= \prod_{r=1}^N \left( \frac{i}{2} \int d\alpha_r d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \frac{i}{\alpha_r E_r - p_r^2 - M_r^2 - 2i\pi_0^{(r)} \bar{\pi}_0^{(r)}} \right) \\ &\times \delta^{OSp} \left( \sum_{s=1}^N p_s^{OSp} \right) F_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}). \end{aligned} \quad (4.8)$$

In  $I_G$ , the moduli concerning the external lines are already integrated. In usual treatment of string theory, the moduli concerning the external lines are taken care of first<sup>6</sup> and string amplitudes are given as integrations over the moduli space of the rest of the worldsheet. In doing so, one tacitly discards contributions from degenerate surfaces which may have some physical significance. For example, masses of massive string states are expected to be shifted by radiative corrections. However such effects are not included in defining the S-matrix of string theory, because the moduli of the external line propagators should be treated simultaneously with the other moduli in order to study these effects. Since we would like to relate our correlation functions to the results of usual formulation of bosonic string theory, we will follow the same order. We will calculate the integrand  $I_G$  for diagrams corresponding to nondegenerate Riemann surfaces.

Let us first show that for  $p_r^2 + M_r^2 \sim 0$ ,  $I_G$  behaves as

$$I_G = \frac{C}{p_r^2 + M_r^2} + \text{less singular terms} . \tag{4.9}$$

Suppose  $r \neq N$  for example. We can integrate over  $\alpha_N, \pi_0^{(N)}, \bar{\pi}_0^{(N)}$  in eq. (4.8) and we obtain

$$\begin{aligned} I_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}) &= \prod_{r=1}^{N-1} \left( \frac{i}{2} \int d\alpha_r d\bar{\pi}_0^{(r)} d\pi_0^{(r)} \frac{i}{\alpha_r E_r - p_r^2 - M_r^2 - 2i\pi_0^{(r)} \bar{\pi}_0^{(r)}} \right) \\ &\times 2\pi \delta \left( \sum_{s=1}^N E_s \right) (2\pi)^{26} \delta^{26} \left( \sum_{s=1}^N p_s \right) F_G \left( p_1^{OSp}, \dots, p_{N-1}^{OSp}, -\sum_{s=1}^{N-1} p_s^{OSp}; \alpha_I, t_n - t_{n-1} \right) \\ &\times \frac{i}{-E_N \sum_{s=1}^{N-1} \alpha_s - p_N^2 - M_N^2 - 2i \left( \sum_{s=1}^{N-1} \pi_0^{(s)} \right) \left( \sum_{s=1}^{N-1} \bar{\pi}_0^{(s)} \right)} . \end{aligned} \tag{4.10}$$

We are interested in the singular behavior of this quantity at  $p_r^2 + M_r^2 = 0$ . Since  $F_G$  is given by a product of factors  $e^{-i\frac{t}{\alpha}(p^2+2i\pi_0\bar{\pi}_0+M^2)}$  and those from the three-string vertices, it cannot be singular at  $p_r^2 + M_r^2 = 0$ . Therefore, for generic momenta  $p_r^\mu$ , such singularities come from the integration over  $\alpha_r$ . For integrating over  $\alpha_r, \pi_0^{(r)}, \bar{\pi}_0^{(r)}$  ( $r = 1, \dots, N-1$ ), we rewrite the propagator again as

$$\begin{aligned} \frac{i}{\alpha_r E_r - p_r^2 - M_r^2 - 2i\pi_0^{(r)} \bar{\pi}_0^{(r)}} &= \int dt e^{iE_r t} \frac{1}{|\alpha_r|} \left[ \theta(\alpha_r) \theta(t) e^{-i\frac{t}{\alpha_r}(p_r^2+2i\pi_0^{(r)}\bar{\pi}_0^{(r)}+M_r^2)} \right. \\ &\left. + \theta(-\alpha_r) \theta(-t) e^{-i\frac{t}{\alpha_r}(p_r^2+2i\pi_0^{(r)}\bar{\pi}_0^{(r)}+M_r^2)} \right] . \end{aligned} \tag{4.11}$$

Thus the integrations we should perform are of the form

$$\frac{i}{2} \int_0^\infty \frac{d\alpha_r}{\alpha_r} \int d\bar{\pi}_0^{(r)} d\pi_0^{(r)} e^{-i\frac{t}{\alpha_r}(p_r^2+2i\pi_0^{(r)}\bar{\pi}_0^{(r)}+M_r^2)} f(\alpha_r, \pi_0^{(r)}, \bar{\pi}_0^{(r)}, p_r) . \tag{4.12}$$

The singular behavior of the integral eq. (4.12) is related to the behavior of the integrand around  $\alpha_r \sim 0$ . If we take  $\alpha_r \rightarrow 0$  keeping the light-cone string diagram  $G$

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<sup>6</sup>Usually we replace them by local vertex operators.

nondegenerate, the  $r$ -th external line can be replaced by a local vertex operator times some factor depending on  $\alpha_r$ . From the form of the three-string vertex, one can deduce

$$F_G(p_r^{OSp}; \alpha_I, t_n - t_{n-1}) = \alpha_r^{p_r^2 + 2i\pi_0^{(r)}\bar{\pi}_0^{(r)} + M_r^2} F_G(p_r^{OSp}; \alpha_I, t_n - t_{n-1}) \Big|_{p_r^2 + M_r^2 = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} + \text{higher order terms} \quad (4.13)$$

for  $\alpha_r \sim 0$ .<sup>7</sup>

Therefore, eq. (4.12) can be evaluated as

$$\frac{1}{p_r^2 + M_r^2} f(\alpha_r, \pi_0^{(r)}, \bar{\pi}_0^{(r)}, p_r) \Big|_{p_r^2 + M_r^2 = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} + \text{less singular terms} . \quad (4.14)$$

Here “less singular terms” indicates terms with less singular behavior at  $p_r^2 + M_r^2 = 0$ . Thus we can see that  $I_G$  has at most a simple pole at  $p_r^2 + M_r^2 = 0$ .

We consider this pole as the one for a particle with mass  $M_r$  and deduce S-matrix elements. In order to do so, we should investigate the most singular part of  $I_G$  for  $p_r^2 + M_r^2 \sim 0$  ( $r = 1, 2, \dots, N$ ). This can be done by successively integrating over  $\alpha_r, \pi_0^{(r)}, \bar{\pi}_0^{(r)}$  for  $r = 1, \dots, N - 1$ . Calculations are essentially the same as above and we eventually get

$$I_G(p_1^{OSp}, \dots, p_N^{OSp}; \alpha_I, t_n - t_{n-1}) = -i \left( \prod_{r=1}^N \frac{2\pi\delta(E_r)}{p_r^2 + M_r^2} \right) (2\pi)^{26} \delta^{26} \left( \sum_{r=1}^N p_r \right) F_G(p_r^{OSp}; \alpha_I, t_n - t_{n-1}) \Big|_{p_r^2 + M_r^2 = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} + \text{less singular terms} . \quad (4.15)$$

On the course of this calculation, we encounter higher order poles of  $p_N^2 + M_N^2$ . These should cancel with each other, because  $I_G$  can have at most simple poles at  $p_r^2 + M_r^2 = 0$  as we have shown above.

## 4.2 S-matrix elements

Substituting eq. (4.15) into eq. (4.8), we formally obtain

$$\left\langle \left\langle \prod_{r=1}^N \tilde{\mathcal{O}}_r(E_r, p_r) \right\rangle \right\rangle = -i \left( \prod_{r=1}^N \frac{2\pi\delta(E_r)}{p_r^2 + M_r^2} \right) (2\pi)^{26} \delta^{26} \left( \sum_{r=1}^N p_r \right) G_{\text{amputated}}^{OSp}(p_r^{OSp}) \Big|_{p_r^2 + M_r^2 = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} + \text{less singular terms} . \quad (4.16)$$

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<sup>7</sup>Here,  $\Big|_{p_r^2 + M_r^2 = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0}$  should be understood so that we put  $p_r^2 + M_r^2$  first and then take the other variables to be 0.

Hence the correlation function for  $\varphi_r(p_r)$  becomes

$$\begin{aligned} \left\langle\left\langle \prod_{r=1}^N \varphi_r(p_r) \right\rangle\right\rangle = & \left( \prod_{r=1}^N \frac{1}{p_r^2 + M_r^2} \right) (-i)(2\pi)^{26} \delta^{26} \left( \sum_{r=1}^N p_r \right) G_{\text{amputated}}^{OSp}(p_r^{OSp}) \Big|_{p_r^2 + M_r^2 = E_r = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} \\ & + \text{less singular terms} . \end{aligned} \tag{4.17}$$

Considering this correlation function as a correlation function of a Euclidean 26 dimensional field theory, we can Wick rotate it and derive the S-matrix element. Let us define the Lorentzian momentum

$$p^L = (p_0, p_1, \dots, p_{25}), \tag{4.18}$$

with  $p_0 = ip_{26}$ . The S-matrix element  $S(p_r^L)$  we obtain is

$$S(p_r^L) = (2\pi)^{26} \delta \left( \sum_{r=1}^N p_r^L \right) G_{\text{amputated}}^{OSp}(p_r^{OSp}) \Big|_{p_r^2 + M_r^2 = E_r = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} . \tag{4.19}$$

We would like to show that this S-matrix element coincides with that in the light-cone gauge string field theory. In order to do so, it is convenient to express  $S(p_r^L)$  by using the S-matrix elements for the  $OSp$  invariant string field theory.

**S-matrix elements for the  $OSp$  invariant string field theory.** If we do not care about the BRST symmetry, the  $OSp$  invariant string field theory is just a light-cone gauge string field theory on a flat supermanifold and the S-matrix elements can be defined in the usual way. Since the  $OSp$  invariant string field theory possesses the  $OSp(27,1|2)$  symmetry, the S-matrix elements are invariant under this symmetry. An on-shell particle state is specified by its momentum  $p^{OSp}$  and polarization  $\epsilon^{OSp}$ . As functions of these variables, the S-matrix elements  $S^{OSp}$  can be expressed as

$$S^{OSp}(p_r^{OSp}, \epsilon_r^{OSp}) = \delta^{OSp} \left( \sum_r p_r^{OSp} \right) f^{OSp}(p^{OSp} \cdot p^{OSp}, p^{OSp} \cdot \epsilon^{OSp}, \epsilon^{OSp} \cdot \epsilon^{OSp}), \tag{4.20}$$

where  $p^{OSp} \cdot p^{OSp}$ ,  $p^{OSp} \cdot \epsilon^{OSp}$  and  $\epsilon^{OSp} \cdot \epsilon^{OSp}$  denote the general invariants of the  $OSp(27,1|2)$  group composed of momenta and polarizations.

By definition,  $f^{OSp}$  coincides with  $G_{\text{amputated}}^{OSp}$  with all the momenta on-shell:

$$f^{OSp} = G_{\text{amputated}}^{OSp} \Big|_{p_r^2 + M_r^2 = 0} . \tag{4.21}$$

The oscillator parts  $|0\rangle_{C, \bar{C}} \otimes |\overline{\text{primary}}_r\rangle_X$  of the states under consideration correspond to polarizations whose components involving  $\pm, C, \bar{C}$  indices are zero. Therefore the last factor on the right hand side of eq. (4.19) can be given by  $f^{OSp}$  with  $p^{OSp}, \epsilon^{OSp}$  whose components involving  $\pm, C, \bar{C}$  indices are zero.

The S-matrix element  $S(p_r^L)$  in eq. (4.19) can be considered as a function of these polarizations. Such polarizations can be considered as tensors in 26 dimensions. Let

us denote the Lorentzian version of these tensors by  $\epsilon^L$ , and the S-matrix element as a function of these polarizations by  $S(p_r^L, \epsilon_r^L)$ . The  $OSp$  invariant combinations  $p^{OSp} \cdot p^{OSp}$ ,  $p^{OSp} \cdot \epsilon^{OSp}$ ,  $\epsilon^{OSp} \cdot \epsilon^{OSp}$  for  $p^{OSp}$ ,  $\epsilon^{OSp}$  satisfying the above-mentioned conditions coincide with the  $SO(25, 1)$  invariant combinations  $p^L \cdot p^L$ ,  $p^L \cdot \epsilon^L$ ,  $\epsilon^L \cdot \epsilon^L$  which are defined in an obvious way following the  $OSp$  version. Hence the S-matrix element  $S(p_r^L, \epsilon_r^L)$  can be written by using eqs. (4.19) and (4.21) as

$$\begin{aligned} S(p_r^L, \epsilon_r^L) &= (2\pi)^{26} \delta^{26} \left( \sum_r p_r^L \right) f^{OSp}(p^{OSp} \cdot p^{OSp}, p^{OSp} \cdot \epsilon^{OSp}, \epsilon^{OSp} \cdot \epsilon^{OSp}) \\ &= (2\pi)^{26} \delta^{26} \left( \sum_r p_r^L \right) f^{OSp}(p^L \cdot p^L, p^L \cdot \epsilon^L, \epsilon^L \cdot \epsilon^L). \end{aligned} \quad (4.22)$$

**Light-cone gauge string field theory.** By construction  $f^{OSp}$  is related to an S-matrix element of the light-cone gauge string field theory. Using this fact, one can deduce from eq. (4.22) that the S-matrix element  $S(p_r^L, \epsilon_r^L)$  coincides with that of the light-cone gauge string field theory, as was done in [3]. Here, for completeness, we will give a proof of this fact.

The light-cone gauge string field theory we have in mind has the action

$$\begin{aligned} \int dt \left[ \frac{1}{2} \int d1_{LC} d2_{LC} {}^{LC} \langle R(1, 2) | \Phi \rangle_1 \left( i \frac{\partial}{\partial t} - \frac{L_0^{LC(2)} + \tilde{L}_0^{LC(2)} - 2}{\alpha_2} \right) | \Phi \rangle_2 \right. \\ \left. + \frac{2g}{3} \int d1_{LC} d2_{LC} d3_{LC} {}^{LC} \langle V_3^0(1, 2, 3) | \Phi \rangle_1 | \Phi \rangle_2 | \Phi \rangle_3 \right], \end{aligned} \quad (4.23)$$

where  $dr_{LC}$  is defined as

$$dr_{LC} \equiv \frac{\alpha_r d\alpha_r}{2} \frac{d^{24} p_r}{(2\pi)^{25}}, \quad (4.24)$$

$L_0^{LC}$  and  $\tilde{L}_0^{LC}$  are the Virasoro generators for the light-cone variables and  ${}^{LC} \langle R(1, 2) |$  and  ${}^{LC} \langle V_3^0(1, 2, 3) |$  are defined in appendix A.

Now let us consider the S-matrix elements of this light-cone gauge string field theory for the external states with momenta and polarizations  $(p_r^L, \epsilon_r^L)$  ( $r = 1, \dots, N$ ). The light-cone gauge string field theory possesses the  $O(25, 1)$  Lorentz symmetry and the S-matrix elements  $S^{LC}$  take the form

$$S^{LC}(p_r^L, \epsilon_r^L) = (2\pi)^{26} \delta^{26} \left( \sum_r p_r^L \right) f^{LC}(p^L \cdot p^L, p^L \cdot \epsilon^L, \epsilon^L \cdot \epsilon^L), \quad (4.25)$$

where  $p^L \cdot p^L$ ,  $p^L \cdot \epsilon^L$  and  $\epsilon^L \cdot \epsilon^L$  denote the general invariants of the  $O(25, 1)$  group.

The  $OSp$  invariant string field theory is constructed from the light-cone gauge string field theory through the  $OSp$  extension. On the worldsheet, the  $OSp$  extension corresponds to adding two bosons  $X^{25}, X^{26}$  and two fermions  $C, \bar{C}$  with spin 0. Therefore it is conceivable that the  $OSp$  extended theory yields the same results as those from the original theory in some situations. Indeed, one can prove that for momenta  $p^{OSp}$  and polarizations  $\epsilon^{OSp}$  whose components involving 25, 26,  $C, \bar{C}$  indices are zero,

$$f^{OSp}(p^{OSp} \cdot p^{OSp}, p^{OSp} \cdot \epsilon^{OSp}, \epsilon^{OSp} \cdot \epsilon^{OSp}) = f^{LC}(p^{OSp} \cdot p^{OSp}, p^{OSp} \cdot \epsilon^{OSp}, \epsilon^{OSp} \cdot \epsilon^{OSp}). \quad (4.26)$$



The proof goes as follows.  $f^{OSp}$  is calculated using the perturbation theory of strings as in eq. (4.6). On the worldsheet we have variables  $X^\mu, C, \bar{C}$ . It is easy to see that the part of the worldsheet theory consisting of bosons  $X^{25}, X^{26}$  and fermions  $C, \bar{C}$  with spin 0 can be considered as a topological field theory. Indeed if we define  $Z = X^{25} + iX^{26}$ , the  $OSp$  generator  $M^{ZC}$  is nilpotent and can be considered as a BRST operator. The Hamiltonian is BRST exact and the three-string vertex is BRST invariant. The states whose momenta and polarizations do not have components involving 25, 26,  $C, \bar{C}$  indices are invariant under this BRST symmetry. Therefore contributions of these variables to the integrand in eq. (4.6) do not depend on the moduli. Thus we calculate them on the surface where  $t_n - t_{n-1} \rightarrow \infty$ . Then we get the factor

$$\lim_{T \rightarrow \infty} \exp \left[ -\frac{T}{|\alpha|} (p_{25}^2 + p_{26}^2 + 2i\pi_0 \bar{\pi}_0) \right] = 2\pi i \delta(p_{25}) \delta(p_{26}) \bar{\pi}_0 \pi_0, \quad (4.27)$$

from the propagator part. Hence only the state  $|0\rangle_{C, \bar{C}} \otimes |0\rangle_{X^{25}, X^{26}}$  with  $p_{25} = p_{26} = \pi_0 = \bar{\pi}_0 = 0$  contribute to the amplitudes. Using these, we can show that the contributions from these variables are 1 and we are left with the light-cone gauge string amplitudes derived from the action eq. (4.23).

The conditions satisfied by the  $p^{OSp}, \epsilon^{OSp}$  in eq. (4.26) is related to those satisfied by  $p^{OSp}, \epsilon^{OSp}$  in eq. (4.22) via Wick rotations and space rotations. Since  $f^{OSp}, f^{LC}$  depend only on the combinations invariant under such manipulations, eq. (4.26) can be used to replace  $f^{OSp}$  in eq. (4.22) by  $f^{LC}$  and we finally obtain

$$S(p_r^L, \epsilon_r^L) = S^{LC}(p_r^L, \epsilon_r^L). \quad (4.28)$$

Before closing this section, one comment is in order. The left hand side of eq. (4.27) can be considered to give a regularization of the delta function on the right hand side, which preserves various symmetries. Using this regularization, we obtain

$$2\pi i \delta(p_{25}) \delta(p_{26}) \bar{\pi}_0 \pi_0 \Big|_{p=\pi_0=\bar{\pi}_0=0} = 1. \quad (4.29)$$

Wick rotating this, we can see that eq. (4.17) can be rewritten as

$$\begin{aligned} \left\langle \left\langle \prod_{r=1}^N \varphi_r(p_r) \right\rangle \right\rangle = & \left( \prod_{r=1}^N \frac{1}{p_r^2 + M_r^2} \right) \left[ \delta^{OSp} \left( \sum_{r=1}^N p_r^{OSp} \right) G_{\text{amputated}}^{OSp}(p_r^{OSp}) \right] \Big|_{p_r^2 + M_r^2 = E_r = \alpha_r = \pi_0^{(r)} = \bar{\pi}_0^{(r)} = 0} \\ & + \text{less singular terms,} \end{aligned} \quad (4.30)$$

in which form the relation to the Parisi-Sourlas reduction may be clearer.

## 5. Discussion

In this paper, we have defined BRST invariant observables in the  $OSp$  invariant string field theory and evaluated correlation functions of them. We have shown that the S-matrix

elements derived from these correlation functions coincide with those of the light-cone gauge string field theory.

Our results will be useful to understand the structure of the  $OSp$  invariant string field theory and explore the proposal in [13] further. The BRST invariant solitonic operators constructed in [13] may be regarded as another kind of observables besides those we constructed in this paper. In this paper, we only care about the particle poles and observables are only BRST invariant up to terms nonlinear in the string fields. Since the boundary states are off-shell states, the observables involving these states should be BRST invariant taking the nonlinear terms into account. What we proposed in [13] are such observables.

The solitonic operators in [13] correspond to D-branes or ghost D-branes [19]. However since we only calculated cylinder amplitudes, we were not able to distinguish the two. Using the method developed in this paper, we may be able to calculate disk amplitudes involving closed string external states, for example. The results depend on whether the soliton is a D-brane or a ghost D-brane and we can fix which of our operators correspond to which.

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### A. Conventions

**$OSp$  invariant string field theory.** The reflector in the  $OSp$  invariant string field theory is given by

$$\langle R(1, 2) | = \delta(1, 2) {}_1\langle 0 | e^{E(1,2)} \frac{1}{\alpha_1}, \tag{A.1}$$

where

$$\begin{aligned} {}_1\langle 0 | &= {}_1\langle 0 | {}_2\langle 0 |, \\ E(1, 2) &= - \sum_{n=1}^{\infty} \frac{1}{n} \left( \alpha_n^{N(1)} \alpha_n^{M(2)} + \tilde{\alpha}_n^{N(1)} \tilde{\alpha}_n^{M(2)} \right) \eta_{NM}, \\ \delta(1, 2) &= 2\delta(\alpha_1 + \alpha_2) (2\pi)^{26} \delta^{26}(p_1 + p_2) i(\bar{\pi}_0^{(1)} + \bar{\pi}_0^{(2)}) (\pi_0^{(1)} + \pi_0^{(2)}). \end{aligned} \tag{A.2}$$

The BPZ conjugate  $\langle \Phi |$  of  $|\Phi\rangle$  is defined as

$${}_2\langle \Phi | = \int d1 \langle R(1, 2) | \Phi \rangle_1. \tag{A.3}$$

The  $*$ -product of the string fields is defined by using

$$\langle V_3(1, 2, 3) | = \delta(1, 2, 3) {}_{123}\langle 0 | e^{E(1,2,3)} C(\rho_I) \mathcal{P}_{123} \frac{|\mu(1, 2, 3)|^2}{\alpha_1 \alpha_2 \alpha_3}, \tag{A.4}$$

where  $\rho_I$  denotes the interaction point and

$$\begin{aligned}
{}_{123}\langle 0| &= {}_1\langle 0| {}_2\langle 0| {}_3\langle 0|, \\
\mathcal{P}_{123} &= \mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3, \quad \mathcal{P}_r = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0^{(r)} - \tilde{L}_0^{(r)})}, \\
\delta(1, 2, 3) &= 2\delta\left(\sum_{s=1}^3 \alpha_s\right) (2\pi)^{26} \delta^{26}\left(\sum_{r=1}^3 p_r\right) i\left(\sum_{r'=1}^3 \bar{\pi}_0^{(r')}\right) \left(\sum_{s'=1}^3 \pi_0^{(s')}\right), \\
E(1, 2, 3) &= \frac{1}{2} \sum_{n,m \geq 0} \sum_{r,s} \bar{N}_{nm}^{rs} \left(\alpha_n^{N(r)} \alpha_m^{M(s)} + \tilde{\alpha}_n^{N(r)} \tilde{\alpha}_m^{M(s)}\right) \eta_{NM}, \\
\mu(1, 2, 3) &= \exp\left(-\hat{\tau}_0 \sum_{r=1}^3 \frac{1}{\alpha_r}\right), \quad \hat{\tau}_0 = \sum_{r=1}^3 \alpha_r \ln |\alpha_r|. \tag{A.5}
\end{aligned}$$

Here  $\bar{N}_{nm}^{rs}$  denote the Neumann coefficients associated with the joining-splitting type of three-string interaction [4–6]. Notice that  $\langle V_3(1, 2, 3)|$  is not equal to the three-string vertex eq. (2.7). The  $*$ -product  $\Phi * \Psi$  of two arbitrary closed string fields  $\Phi$  and  $\Psi$  is given as

$$|\Phi * \Psi\rangle_4 = \int d1d2d3 \langle V_3(1, 2, 3)|\Phi\rangle_1 |\Psi\rangle_2 |R(3, 4)\rangle. \tag{A.6}$$

**Light-cone gauge string field theory.** Various quantities appearing in the light-cone gauge string field theory action can be defined in a quite similar way. The reflector  ${}^{\text{LC}}\langle R(1, 2)|$  is given by

$${}^{\text{LC}}\langle R(1, 2)| = \delta_{\text{LC}}(1, 2) {}_{12}\langle 0| e^{E_{\text{LC}}(1,2)} \frac{1}{\alpha_1}, \tag{A.7}$$

where

$$\begin{aligned}
{}_{12}\langle 0| &= {}_1\langle 0| {}_2\langle 0|, \\
E_{\text{LC}}(1, 2) &= -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{24} \left(\alpha_n^{i(1)} \alpha_n^{i(2)} + \tilde{\alpha}_n^{i(1)} \tilde{\alpha}_n^{i(2)}\right), \\
\delta_{\text{LC}}(1, 2) &= 2\delta(\alpha_1 + \alpha_2) (2\pi)^{25} \delta^{24}(p_1 + p_2). \tag{A.8}
\end{aligned}$$

The three-string vertex  ${}^{\text{LC}}\langle V_3^0(1, 2, 3)|$  can be defined as

$${}^{\text{LC}}\langle V_3^0(1, 2, 3)| = \delta_{\text{LC}}(1, 2, 3) {}_{123}\langle 0| e^{E_{\text{LC}}(1,2,3)} \mathcal{P}_{123}^{\text{LC}} \frac{|\mu(1, 2, 3)|^2}{\alpha_1 \alpha_2 \alpha_3}, \tag{A.9}$$

where

$$\begin{aligned}
{}_{123}\langle 0| &= {}_1\langle 0| {}_2\langle 0| {}_3\langle 0|, \\
\mathcal{P}_{123}^{\text{LC}} &= \mathcal{P}_1^{\text{LC}} \mathcal{P}_2^{\text{LC}} \mathcal{P}_3^{\text{LC}}, \quad \mathcal{P}_r^{\text{LC}} = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta(L_0^{\text{LC}(r)} - \tilde{L}_0^{\text{LC}(r)})}, \\
\delta_{\text{LC}}(1, 2, 3) &= 2\delta\left(\sum_{s=1}^3 \alpha_s\right) (2\pi)^{25} \delta^{24}\left(\sum_{r=1}^3 p_r\right), \\
E_{\text{LC}}(1, 2, 3) &= \frac{1}{2} \sum_{n,m \geq 0} \sum_{r,s} \sum_{i=1}^{24} \bar{N}_{nm}^{rs} \left(\alpha_n^{i(r)} \alpha_m^{i(s)} + \tilde{\alpha}_n^{i(r)} \tilde{\alpha}_m^{i(s)}\right), \\
\mu(1, 2, 3) &= \exp\left(-\hat{\tau}_0 \sum_{r=1}^3 \frac{1}{\alpha_r}\right), \quad \hat{\tau}_0 = \sum_{r=1}^3 \alpha_r \ln |\alpha_r|. \tag{A.10}
\end{aligned}$$

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